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NONINTEGRABILITY OF A FIFTH-ORDER EQUATION WITH INTEGRABLE TWO-BODY DYNAMICS

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We consider a fifth-order partial differential equation (PDE) that is a generalization of the integrable Camassa–Holm equation. This fifth-order PDE has exact solutions in terms of an arbitrary number of superposed pulsons with a geodesic Hamiltonian dynamics that is known to be integrable in the two-body case $N = 2$. Numerical simulations show that the pulsons are stable, dominate the initial value problem, and scatter elastically. These characteristics are reminiscent of solitons in integrable systems. But after demonstrating the nonexistence of a suitable Lagrangian or bi-Hamiltonian structure and obtaining negative results from Painlevé analysis and the Wahlquist–Estabrook method, we assert that this fifth-order PDE is not integrable.

Keywords: Hamiltonian dynamics, nonintegrability, elastic scattering, pulsons

1. Introduction

This note is concerned with the fifth-order partial differential equation (PDE)

$$u_{4x,t} - 5u_{xxt} + 4u_t + uu_{5x} + 2u_x u_{4x} - 5uu_{3x} - 10u_x u_{xx} + 12uu_x = 0. \tag{1.1}$$

One reason for our interest in this equation is that it admits exact solutions of the form

$$u = \sum_{j=1}^{N} p_j(t)\big(2e^{-|x-q_j(t)|} - e^{-2|x-q_j(t)|}\big), \tag{1.2}$$

where p_j and q_j satisfy the canonical Hamiltonian dynamics generated by

$$H_N = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k \big(2e^{-|q_j-q_k|} - e^{-2|q_j-q_k|}\big). \tag{1.3}$$

Following [1], we call such solutions "pulsons." The equations for the N-body pulson dynamics are equivalent to a geodesic flow on an N-dimensional space with the coordinates q_j and the cometric

$$g^{jk} = g(q_j - q_k), \qquad g(x) = 2e^{-|x|} - e^{-2|x|}.$$

Pulsons (1.2) are weak solutions with discontinuous second derivatives at isolated points.

Partial differential equation (1.1) is one of a family of integral PDEs, considered in [1], given by

$$m_t + um_x + 2u_x m = 0, \qquad u = g * m, \tag{1.4}$$

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where $u(x,t)$ is defined in terms of $m(x,t)$ by the convolution integral

$$g * m := \int_{-\infty}^{\infty} g(x-y) m(y,t)\, dy.$$

The integral kernel $g(x)$ is taken to be an even function, and for any g, Eq. (1.4) has the Lie–Poisson Hamiltonian form

$$m_t = -(m\partial_x + \partial_x m)\frac{\delta H}{\delta m}, \tag{1.5}$$

where

$$H = \frac{1}{2}\int m\, g * m\, dx = \frac{1}{2}\int mu\, dx. \tag{1.6}$$

Any equation in this family admits pulson solutions

$$u(x,t) = \sum_{j=1}^{N} p_j(t) g(x - q_j(t))$$

for arbitrary N, where p_j and q_j satisfy the canonical Hamilton equations

$$\frac{dp_j}{dt} = -\frac{\partial H_N}{\partial q_j} = -p_j \sum_{k=1}^{N} p_k g'(q_j - q_k), \qquad \frac{dq_j}{dt} = \frac{\partial H_N}{\partial p_j} = \sum_{k=1}^{N} p_k g(q_j - q_k), \tag{1.7}$$

generated by the Hamiltonian

$$H_N = \frac{1}{2}\sum_{j,k} p_j p_k\, g(q_j - q_k).$$

Equations (1.7) correspond to geodesic motion on a manifold with the cometric $g^{jk} = g(q_j - q_k)$. A significant result in [1] is that the two-body (N=2) dynamics is integrable for any choice of the kernel g, and numerical calculations show that this elastic two-pulson scattering dominates the initial value problem.

Three special cases are isolated in [1], namely (up to suitable scaling),

- $g(x) = \delta(x)$ – Riemann shocks,
- $g(x) = 1 - |x|, \ |x| < 1$ – compactons,
- $g(x) = e^{-|x|}$ – peakons.

For each of these cases, both integral PDE (1.4) and the corresponding finite-dimensional system (1.7) (for any N) are integrable. Most relevant here is the third case, where $g(x) = e^{-|x|}$, which is the (scaled) Green's function for the Helmholtz operator, satisfying the identity

$$(1 - \partial_x^2) g(x) = 2\delta(x).$$

After rescaling in this case, we can take

$$m = u - u_{xx}, \tag{1.8}$$

and Eq. (1.4) is just a PDE for $u(x,t)$, namely,

$$u_t - u_{xxt} - uu_{3x} - 2u_x u_{xx} + 3uu_x = 0, \tag{1.9}$$

which is the dispersionless form of the integrable Camassa–Holm equation for shallow water waves [2]. For the Camassa–Holm equation, the pulson solutions take the form of peakons or peaked solitons, i.e.,

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}. \tag{1.10}$$

The fifth-order equation arises from a different choice of the Green's function. Using the identity

$$(4-\partial_x^2)(1-\partial_x^2)g(x) = 12\delta(x), \qquad g(x) = 2e^{-|x|} - e^{-2|x|}, \tag{1.11}$$

we find that this choice of g (after suitable scaling) yields

$$m = u_{4x} - 5u_{xx} + 4u, \tag{1.12}$$

and then Eq. (1.4) becomes fifth-order equation (1.1). Therefore, PDE (1.1) should be considered a natural higher-order generalization of the Camassa–Holm equation.

2. Hamiltonian and Lagrangian considerations

In a forthcoming article [3], we discuss a more general family of integral PDEs of the form

$$m_t + um_x + bu_x m = 0, \qquad u = g * m, \tag{2.1}$$

where b is an arbitrary parameter; family (1.4) corresponds to the particular case $b = 2$; and numerical results were recently established for different b values in [4]. In the case of the peakon kernel $g = e^{-|x|}$ with m given by (1.8), the equations in this class were tested by the method of asymptotic integrability [5], and only the cases $b = 2, 3$ were isolated as potentially integrable. For $b = 2$, the integrability of the Camassa–Holm equation by inverse scattering was already known [2], but for the new equation

$$u_t - u_{xxt} - uu_{3x} - 3u_x u_{xx} + 4uu_x = 0 \tag{2.2}$$

with $b = 3$, the integrability was proved in [6] by constructing the Lax pair. The two integrable cases $b = 2, 3$ were also found recently via the perturbative symmetry approach [7]. For any $b \neq -1$, peakon family (2.1) with $g = e^{-|x|}$ arises as the dispersionless limit at the quadratic order in the asymptotic expansion for shallow water waves [8].

Another motivation for our interest in fifth-order equation (1.1) is that it is naturally expressed in terms of the quantity m (Eq. (1.12)) given by the product of two Helmholtz operators acting on u. Such a product appears in the fifth-order operator

$$B_0 = \partial_x(4-\partial_x^2)(1-\partial_x^2),$$

which we found [6] provides the first Hamiltonian structure for new integrable equation (2.2). This led us to conjecture that the operator B_0 should appear naturally in the theory of higher-order integrable equations such as (1.1).

All equations of form (1.4) have the Lie–Poisson Hamiltonian structure given by (1.5), but for integrability, we expect a bi-Hamiltonian structure. In the case of Camassa–Holm equation (1.9), there are two ways to derive a second Hamiltonian structure. The first is by inspection using a conservation law, noting that (1.9) can be written as

$$m_t = \left(uu_{xx} + \frac{1}{2}u_x^2 - \frac{3}{2}u^2\right)_x = \partial_x \frac{\delta \widetilde{H}}{\delta u} = \partial_x(1-\partial_x^2)\frac{\delta \widetilde{H}}{\delta m} \tag{2.3}$$

for

$$\widetilde{H} = -\frac{1}{2}\int (uu_x^2 + u^3)\, dx. \tag{2.4}$$

Identity (2.3) gives the second Hamiltonian structure for the Camassa–Holm equation, and $m\partial_x + \partial_x m$ and $\partial_x(1-\partial_x^2)$ constitute a compatible bi-Hamiltonian pair.

Similarly, $\int m\, dx$ is conserved for (1.1) with m given by (1.12). The explicit conservation law is

$$(u_{4x} - 5u_{xx} + 4u)_t = -\left(uu_{4x} + u_x u_{3x} - \frac{1}{2}u_{xx}^2 - 5uu_{xx} - \frac{5}{2}u_x^2 + 6u^2\right)_x =: \mathcal{F}_x. \tag{2.5}$$

By analogy with the Camassa–Holm equation, this would suggest that a suitable constant-coefficient Hamiltonian operator might be $\partial_x(4-\partial_x^2)(1-\partial_x^2)$ (which we know to be a Hamiltonian operator for the new equation, Eq. (2.2)). This would require the right-hand side of (2.5) to take the form

$$\partial_x \frac{\delta K}{\delta u} = \partial_x(4-\partial_x^2)(1-\partial_x^2)\frac{\delta K}{\delta m}.$$

But for the flux of (2.5), we find

$$\mathcal{F} \neq \frac{\delta K}{\delta u}$$

for any local density functional K of u, and we suppose that the operators $\partial_x(4-\partial_x^2)(1-\partial_x^2)$ and $m\partial_x + \partial_x m$ must be incompatible.

The second way to derive Hamiltonian structure (2.3) for the Camassa–Holm equation is via the action (integral of the Lagrangian density)

$$S = \iint \mathcal{L}[\phi]\, dx\, dt := \iint \frac{1}{2}(\phi_x\phi_t - \phi_{3x}\phi_t + \phi_x\phi_{xx}^2 + \phi_x^3)\, dx\, dt$$

for $u = \phi_x$. A Legendre transformation yields the conjugate momentum

$$\frac{\partial \mathcal{L}}{\partial \phi_t} = \frac{1}{2}(\phi_x - \phi_{3x}) = \frac{m}{2}$$

and the same Hamiltonian as (2.4) above, i.e.,

$$\widetilde{H} = \int \left(\frac{1}{2}m\phi_t - \mathcal{L}\right) dx.$$

Trying the same approach for (1.1), we set $u = \phi_x$ and rewrite it as

$$\phi_{5x,t} - 5\phi_{3x,t} + 4\phi_{xt} + \phi_x\phi_{6x} + 2\phi_{xx}\phi_{5x} - 5\phi_x\phi_{4x} - 10\phi_{xx}\phi_{3x} + 12\phi_x\phi_{xx} = 0. \tag{2.6}$$

But Eq. (2.6) cannot be derived from a local Lagrangian density $\mathcal{L}[\phi]$ because of the presence of the terms $\phi_x\phi_{6x} + 2\phi_{xx}\phi_{5x}$.

The first nonlocal Hamiltonian structure for the Camassa–Holm equation is obtained by applying the recursion operator to $m\partial_x + \partial_x m$. This means that (1.9) can be written in the Hamiltonian form

$$m_t = (m\partial_x + \partial_x m)(\partial_x^3 - \partial_x)^{-1}(m\partial_x + \partial_x m)\frac{\delta \widehat{H}}{\delta m}, \qquad \widehat{H} = \int m\, dx.$$

With the same $\widehat{H}$, the analogous identity for (1.1) is

$$m_t = B\frac{\delta \widehat{H}}{\delta m} \equiv (m\partial_x + \partial_x m)(\partial_x^5 - 5\partial_x^3 + 4\partial_x)^{-1}(m\partial_x + \partial_x m)\frac{\delta \widehat{H}}{\delta m}, \tag{2.7}$$

but we would expect from the above considerations that the formal nonlocal operator B in the right-hand side of (2.7) is not Hamiltonian; indeed, using the functional equations derived in [3], we can show that it fails to satisfy the Jacobi identity.

3. Reciprocal transformation and Painlevé analysis

Having failed to find the sort of Lagrangian or bi-Hamiltonian structure for (1.1) that we would reasonably expect, we proceed to see what Painlevé analysis can tell us about this fifth-order equation. But we note that both Camassa–Holm equation (1.9) and the new equation, Eq. (2.2), provide examples of the weak Painlevé property [9] with algebraic branching in the solutions. For these equations, we found it convenient to use reciprocal transformations (see [10] for definitions), which transform to equations with pole singularities; indeed, this was the key to our discovery of the Lax pair for (2.2) in [6]. Hodograph transformations of this kind have been used before to remove branching from classes of evolution equations [11], [12], but we are dealing with nonevolution PDEs here.

To make the results of our analysis more general, we consider the whole class of equations

$$m_t + um_x + bu_x m = 0, \qquad m = u_{4x} - 5u_{xx} + 4u \tag{3.1}$$

for arbitrary nonzero b, which is the particular family of Eqs. (2.1) corresponding to integral kernel (1.11) and includes (1.1) as the special case $b = 2$. Each equation in class (3.1) has the conservation law

$$(m^{1/b})_t = -(m^{1/b}u)_x.$$

Therefore, introducing a new dependent variable p according to

$$p^b = -m, \tag{3.2}$$

we can consistently define a reciprocal transformation to new independent variables X and T given by

$$dX = p\, dx - pu\, dt, \qquad dT = dt. \tag{3.3}$$

Transforming the derivatives, we have the new conservation law

$$(p^{-1})_T = u_X. \tag{3.4}$$

Rewriting relation (1.12) in terms of ∂_X and using (3.4) to eliminate derivatives of u, we obtain the identity

$$u = \frac{1}{4}(5 - (p\partial_X)^2)(p\partial_X)p(p^{-1})_T - \frac{p^b}{4}, \tag{3.5}$$

which means that (3.4) can be written as an equation for p alone, i.e.,

$$(p^{-1})_T = \left(\frac{1}{4}((p\partial_X)^2 - 5)p(\log p)_{XT} - \frac{p^b}{4}\right)_X. \tag{3.6}$$

Fifth-order equation (3.6) is the reciprocal transform of (3.1). Rather than performing the full Painlevé test for the transformed equation, it suffices for our purposes to follow [13] and apply the Painlevé test for ODEs to the traveling-wave reduction of (3.6). Hence, we set $p = p(z)$ and $z = X - cT$. The resulting fifth-order ODE can be integrated twice to obtain the third-order ODE

$$\frac{5}{8}\left(\frac{p'}{p}\right)^2 - \frac{1}{4}\left(p'p''' - \frac{1}{2}(p'')^2 - \frac{(p')^2p''}{p} + \frac{1}{2}\frac{(p')^4}{p^2}\right) - \frac{1}{2p^2} = \frac{c^{-1}p^{b-1}}{4(b-1)} + \frac{d}{p} + e, \tag{3.7}$$

$b \neq 1$, where d and e are arbitrary constants, c is the wave speed, and the prime denotes d/dz. For $b = 1$, there is a $\log p$ term on the right-hand side, and this case therefore has logarithmic branching and is immediately excluded by the Painlevé test. Similarly, because of the p^{b-1} term all noninteger values of b have branching and are discarded.

We proceed to apply Painlevé analysis to (3.7) for integer $b \neq 0, 1$, seeking the leading-order behavior at a movable point z_0 of the form $p \sim a(z - z_0)^\mu$ for an integer exponent μ. For all integers $b \leq -2$, the only possible balance is $\mu = 4/(3 - b)$, which is noninteger and hence gives algebraic branching. In the special case $b = -1$, there are four possible balances with $\mu = 1$, where a and the resonances depend on c. We verified that no value of c gives all integer resonances, and the Painlevé test is therefore failed. For the remaining cases of integer $b \geq 2$, we find $\mu = 1$ with $a^2 = 1$ or $a^2 = 4$. For integer $b \geq 4$, there is also the balance $\mu = 4/(3 - b)$, which is generally noninteger, except for the special cases $\mu = -4, -2, -1$ for $b = 4, 5, 7$ respectively. Therefore, all integer values of b except $b = 2, 3, 4, 5, 7$ are ruled out by the (strong) Painlevé test because of algebraic branching. But we could still use the weak Painlevé test if we allow such branching.

We consider the first two types of balance for integer $b \geq 2$ in more detail. When $p \sim \pm(z - z_0)$, we have a nonprincipal balance with the resonances $r = -1, -1, 3$. Interestingly, for $b = 2$, the resonance condition at $r = 3$ is failed, the obstruction being the c^{-1} term (for these balances, the test is hence passed only in the limit $c \to \infty$), but for all integer $b \geq 3$, it is satisfied. But for the principal balances $p \sim \pm 2(z - z_0)$, the resonances are $r = -1, 1/2, 3/2$, which means that there is algebraic branching and the (strong) Painlevé test is hence failed for any b. We further checked whether the weak Painlevé test in [9] could be satisfied by allowing an expansion in powers of $(z - z_0)^{1/2}$ in the principal balance. The resonance condition is satisfied at $r = 1/2$ but is failed at $r = 3/2$, which means that this expansion with square-root branching cannot represent the general solution, because it does not contain enough arbitrary constants. The arbitrariness can only be restored by adding infinitely many terms in powers of $\log(z - z_0)$; therefore, no form of the Painlevé property can be recovered. The existence of logarithmic branching in both the principal and nonprincipal balances is a strong indication of nonintegrability.

It is interesting to observe that when the first term in the right-hand side of (3.7) is absent (the limit $c \to \infty$), it admits exact solutions in terms of trigonometric/hyperbolic functions, corresponding to the first-order reductions

$$(p')^2 = 1 + 2dp + \frac{1}{3}(8e - d^2)p^2, \qquad (p')^2 = 4 + 8dp + \frac{8}{3}(2d^2 - e)p^2.$$

In fact, we can also see that original equation (1.1) fails the weak Painlevé test directly. For Camassa–Holm equation (1.9), the test is satisfied by a principal balance

$$u \sim -\frac{\phi_t}{\phi_x} + a\phi^{2/3} + \cdots$$

with the resonances $-1, 0, 2/3$ with the singular manifold $\phi(x,t)$ and $a(x,t)$ being arbitrary. For (1.1), there is an analogous balance

$$u \sim -\frac{\phi_t}{\phi_x} + a\phi^{4/3} + \cdots$$

with resonances $-1, 0, 4/3, (1 \pm \sqrt{41})/6$; the presence of irrational resonances implies logarithmic branching.

4. Prolongation algebra method

While Painlevé analysis is a good heuristic tool for isolating potentially integrable equations, it can never be said to provide definite proof of nonintegrability. If a precise definition of integrability is given in terms of the existence of infinitely many commuting symmetries, then the symmetry approach of Shabat et al. [14] gives necessary conditions for integrability (but provides neither a constructive way to find a Lax pair nor linearization when the conditions are satisfied). The symmetry approach has only very recently been extended [7] such that it can be applied to nonlocal or nonevolution equations, such as (1.1) and (1.9). As an alternative, we use the prolongation algebra method of Wahlquist and Estabrook [15] and directly seek a Lax pair for (1.1) in the form of a compatible linear system

$$\Psi_x = U\Psi, \qquad \Psi_t = V\Psi \tag{4.1}$$

for suitable matrices U and V (usually taking values in the fundamental representation of a semisimple Lie algebra), which should depend on u, its derivatives, and a spectral parameter. We found the clear presentation of the method in [16] very useful.

The compatibility of system (4.1) yields the zero-curvature equation

$$U_t - V_x + [U, V] = 0, \tag{4.2}$$

and the essence of the Wahlquist–Estabrook method is that given the original PDE (Eq. (1.1) in this case), Eq. (4.2) can be used to derive the functional dependence of U and V on u, u_x, etc. A negative result means that no Lax pair of a suitable form exists, suggesting that the equation is not integrable, but this, of course, is sensitive to the initial assumptions about the functional form of U and V.

For ease of notation, we write the nth derivative $u_{nx} = u_n$. Given that (1.1) can be written as a conservation law for m as in (2.5), a reasonable ansatz is to assume that

$$U = U(m), \qquad V = V(u, u_1, u_2, u_3, u_4)$$

(with the dependence on the spectral parameter suppressed). Given the known form of the zero-curvature representations for Eqs. (1.9) and (2.2), we further assume that U is linear in m and therefore

$$U = Am + B \equiv (u_4 - 5u_2 + 4u)A + B,$$

where A and B are constant matrices (independent of x and t but potentially dependent on the spectral parameter). Substituting this ansatz in (4.2) and using (2.5) to eliminate the t derivative m_t, we find

$$\begin{aligned}(-uu_5 - 2u_1u_4 + 5uu_3 + 10u_1u_2 - 12uu_1)A - u_5V_{u_4} - u_4V_{u_3} -& \\ - u_3V_{u_2} - u_2V_{u_1} - u_1V_u + (u_4 - 5u_2 + 4u)[A, V] + [B, V] = 0,&\end{aligned} \tag{4.3}$$

where the subscripts on V denote partial derivatives. None of the matrices depend on u_5, and (4.3) is therefore linear in u_5. In particular, the coefficient of u_5 must vanish, giving the equation $V_{u_4} = -uA$, which integrates immediately giving

$$V = -uu_4A + \Gamma(u, u_1, u_2, u_3), \tag{4.4}$$

where Γ is so far arbitrary and must be determined from the remaining terms in (4.3).

At the next step, we substitute (4.4) in (4.3) and obtain

$$\begin{aligned}(-u_1u_4 + 5uu_3 + 10u_1u_2 - 12uu_1)A - u_4\Gamma_{u_3} - u_3\Gamma_{u_2} - u_2\Gamma_{u_1} - \\ - u_1\Gamma_u + (u_4 - 5u_2 + 4u)[A, \Gamma] + uu_4[A, B] + [B, \Gamma] = 0.\end{aligned} \tag{4.5}$$

The coefficient of u_4 gives the equation

$$\Gamma_{u_3} = [A, \Gamma + uB] - u_1A,$$

which can be integrated exactly as

$$\Gamma = e^{u_3A}\Delta(u, u_1, u_2)e^{-u_3A} - u_1u_3A - uB, \tag{4.6}$$

where Δ is the arbitrary function of integration. From (4.6), we see the presence of $\mathrm{Ad}\exp u_3A = \exp(\mathrm{ad}\, u_3A)$ acting on Δ, which would imply exponential-type dependence on u_3 in the Lax pair unless $(\mathrm{ad}\, u_3A)^n\Delta = 0$ for some positive integer n. Such exponential dependence would seem unlikely given that original equation (1.1) is polynomial in u and its derivatives, and we seek assumptions that prohibit infinitely many nonzero commutators occurring in (4.6).

Substituting Γ given by (4.6) in the u_4-independent terms in (4.5) and applying $\mathrm{Ad}\exp(-u_3A)$, we obtain

$$\begin{aligned}(u_2u_3 + 5uu_3 + 10u_1u_2 - 12uu_1)A - u_3\Delta_{u_2} - u_2\Delta_{u_1} - u_1\Delta_u + \\ + (-5u_2 + 4u)[A, \Delta] + [e^{-u_3A}Be^{u_3A}, \Delta] + \\ + (u_1u_3 + 5uu_2 - 4u^2)e^{-u_3A}Ce^{u_3A} = 0,\end{aligned} \tag{4.7}$$

where we set $C = [A, B]$. Potentially, (4.7) is an infinite power series in u_3, each coefficient of which must vanish. The simplest assumption we can make to terminate the series is to take

$$[A, C] = 0, \tag{4.8}$$

which implies

$$\mathrm{Ad}\, e^{-u_3A}(B) = e^{\mathrm{ad}(-u_3A)}(B) = B - u_3C,$$
$$\mathrm{Ad}\, e^{-u_3A}(C) = e^{\mathrm{ad}(-u_3A)}(C) = C,$$

and (4.7) hence becomes linear in u_3. A fortunate consequence of (4.8) is that the coefficient of u_3 gives

$$\Delta_{u_2} = (u_2 + 5u)A - [C, \Delta] + u_1C,$$

which integrates exactly without further assumptions to yield

$$\Delta = e^{-u_2 C} E(u, u_1) e^{u_2 C} + \left(\frac{1}{2}u_2^2 + 5uu_2\right) A + u_1 u_2 C. \tag{4.9}$$

After acting with $\operatorname{Ad}\exp u_2 C$, we obtain the remaining terms in (4.7) in the form

$$\begin{aligned}(5u_1u_2 - 12uu_1)A - \left(\frac{3}{2}u_2^2 + 4u^2\right) C - u_2 E_{u_1} - u_1 E_u + \\ + (-5u_2 + 4u)[A, E] + [e^{u_2 C} B e^{-u_2 C}, E] + u_1 u_2 [e^{u_2 C} B e^{-u_2 C}, C] = 0.\end{aligned} \tag{4.10}$$

We are again faced with an infinite power series, this time in u_2. Before seeking further simplifying assumptions, we note that the coefficient of the term linear in u_2 is just

$$-E_{u_1} + u_1(5A + [B, C]) - [5A + [B, C], E] = 0,$$

which integrates immediately to

$$E = e^{-u_1 D} Z(u) e^{u_1 D} + \frac{1}{2}u_1^2 D, \qquad D = 5A + [B, C]. \tag{4.11}$$

To analyse the other terms in (4.10), we find it convenient to introduce the quantities

$$F = [C, [C, B]], \qquad G = [D, B].$$

We note that the identities

$$[A, D] = 0 = [A, F], \qquad [A, G] = -[C, D] = F, \qquad [[B, C], D] = 0 \tag{4.12}$$

all hold.

The coefficient of u_2^2 in (4.10) is then

$$-\frac{3}{2}C + \frac{1}{2}[F, E] - u_1 F = 0, \tag{4.13}$$

and for the coefficient of u_2^0, we obtain

$$-12uu_1 A - 4u^2 e^{u_1 D} C e^{-u_1 D} - u_1 Z_u + 4u[A, Z] + \left[e^{u_1 D} B e^{-u_1 D}, Z + \frac{1}{2}u_1^2 D\right] = 0 \tag{4.14}$$

(after substituting E from (4.11) and acting with $\operatorname{Ad} e^{u_1 D}$). We do not need to consider the equations

$$[(\operatorname{ad} C)^n B, E] - n u_1 (\operatorname{ad} C)^n B = 0,$$

occurring at u_2^n, $n \geq 3$. Instead, we consider the coefficient of u_1 in (4.14), which is

$$-Z_u + [G, Z] - 4u^2 F - 12uA = 0. \tag{4.15}$$

We are unable to integrate this directly without making a further assumption, the simplest possible being

$$[F, G] = 0, \tag{4.16}$$

which implies

$$Z = e^{Gu} \Theta e^{-Gu} + \frac{2}{3} u^3 F - 6u^2 A. \tag{4.17}$$

We must now use the remaining equations to determine the commutation relations for the constant Lie algebra elements A, B, C, D, F, G, and Θ. Considering the coefficient of u_1^0 in (4.13), we see that

$$-\frac{3}{2} C + \frac{1}{2} [F, Z] = 0,$$

and using (4.12) and (4.16) with (4.17) yields

$$[F, \Theta] = 3 e^{-Gu} C e^{Gu},$$

which immediately implies

$$[F, \Theta] = 3C, \qquad [C, G] = 0. \tag{4.18}$$

Using the Jacobi identity, we also have

$$0 = [C, G] = [C, [D, B]] = -[B, [C, D]] - [D, [B, C]] = [B, F],$$

which by further applications of the Jacobi identity gives

$$[C, F] = 0 = [D, F]. \tag{4.19}$$

We now return to Eq. (4.13) and use (4.11) to evaluate the coefficient of u_1 as

$$-\frac{1}{2} [F, [D, Z]] - F = 0. \tag{4.20}$$

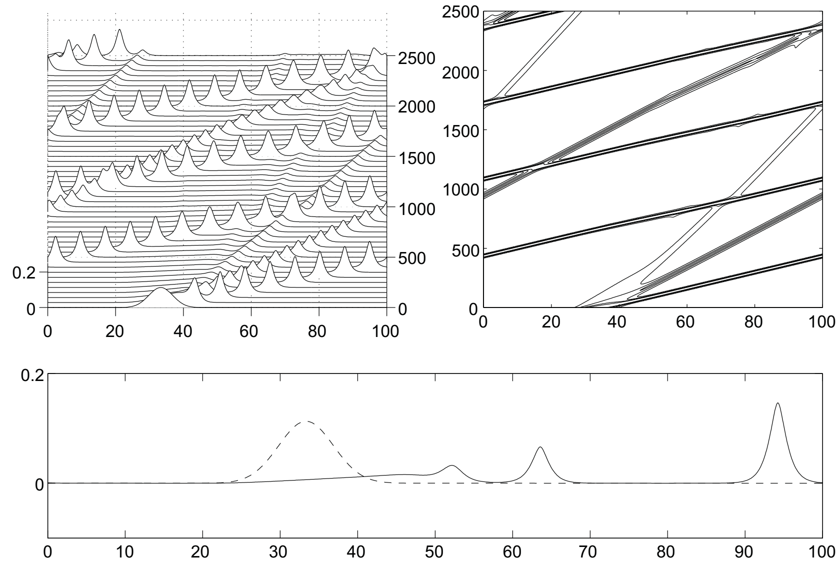

Fig. 1. Pulson solutions (1.2) of Eq. (1.1) emerge from a Gaussian of unit area and width $\sigma = 5$ centered about $x = 33$ on a periodic domain of length $L = 100$. The fastest pulson crosses the domain four times and collides elastically with the slower ones.

Substituting Z given by (4.17) and taking the constant coefficient u^0 in (4.20), we use (4.12), (4.18), and (4.19) to find

$$0 = -\frac{1}{2}[F, [D, \Theta]] - F = \frac{1}{2}\big([\Theta, [F, D]] + [D, [\Theta, F]]\big) - F =$$

$$= -\frac{3}{2}[D, C] - F = -\frac{5}{2}F.$$

Then $F = 0$ implies $C = 0$ from the first equation in (4.18), and it is straightforward to show that Lax pair (4.1) collapses down to the trivial case $[U, V] = 0$, with (4.2) reducing to the scalar equation $m_t = \mathcal{F}_x$ as in (2.5).

5. Conclusions

Because fifth-order PDE (1.1) is in the class of pulson equations studied by Fringer and Holm [1], it admits exact solutions in the form of a direct superposition of an arbitrary number of pulsons (as in Fig. 1). These particular N-pulson solutions have the precise form (1.2), and by the general results in [1], we know that for $N = 2$, Hamiltonian equations (1.7) describing the two-body dynamics are integrable. But several different considerations provide strong evidence that fifth-order PDE (1.1) is not integrable in the sense of admitting a Lax pair and being solvable by the inverse scattering transform.

It is well known that integrable PDEs, such as the Korteweg–de Vries or Camassa–Holm equations [2], admit a compatible pair of Hamiltonian structures that together define a recursion operator generating infinitely many higher symmetries. We tried and failed to find an analogous bi-Hamiltonian or Lagrangian formulation for fifth-order equation (1.1); as far as we are aware, it admits only the single Hamiltonian structure (1.5).

Both Camassa–Holm equation (1.9) and new integrable equation, Eq. (2.2), isolated by Degasperis and Procesi [5], exhibit the weak Painlevé property in [9] with algebraic branching in local expansions representing a general solution. There are many examples of Liouville integrable systems in finite dimensions [17] and Lax-integrable PDEs [12] with this property. For evolution equations, transformations of

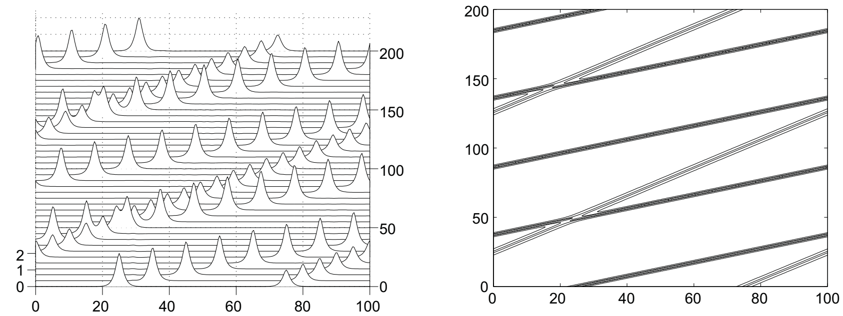

Fig. 2. Two rear-end collisions of pulson solutions (1.2) of Eq. (1.1) with the initial positions $x = 25$ and $x = 75$: The faster pulson moves at twice the speed of the slower one. For this ratio of speeds, both collisions result in a phase shift to the right for the faster space–time trajectory but no phase shift for the slower one.

the hodograph type can restore the strong Painlevé property [11], [12], and similarly in [3], [6], we used reciprocal transformations for nonevolution equations (1.9) and (2.2). To apply Painlevé analysis to (1.1), we found it convenient to apply a reciprocal transformation that removes the branching at the leading order, but further analysis of the traveling-wave reduction shows that there is still algebraic branching in the principal balances due to half-integer resonances. Furthermore, in both the principal and the nonprincipal balances, a resonance condition is failed; therefore, even the weak Painlevé test cannot be satisfied after the transformation.

We have also applied an integrability test that is perhaps less fashionable nowadays, namely, the prolongation algebra method of Wahlquist and Estabrook [15]. By making certain simple assumptions, we find that no polynomial Lax pair of a suitable form exists for (1.1).

We cannot expect the N-body pulson dynamical system to be integrable for arbitrary $N > 2$, because this would imply the existence of an infinite integrable subsector within a nonintegrable PDE. But it would be good to find an analytic explanation for the apparent soliton-like behavior of the pulson solutions and their numerical stability as evidenced in Fig. 2. Further analytic and numerical studies will be needed to understand the stability properties of the pulson solutions.

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